Global exponential stability of hybrid bidirectional associative memory neural networks with discrete delays

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In this paper, the dynamical characteristics of hybrid bidirectional associative memory neural networks with constant transmission delays are investigated. Without assuming symmetry of synaptic connection weights and monotonicity and differentiability of activation functions, Halanay-type inequalities (which are different from the approach of constructing Lyapunov functionals) are employed to derive the delay-independent sufficient conditions under which the networks converge exponentially to the equilibria associated with temporally uniform external inputs. Our results are less conservative and restrictive than previously known results.

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I. INTRODUCTION

During recent decades, a class of neural networks related to bidirectional associative memory (BAM) has been proposed [1]. These models generalized the single-layer autoassociative Hebbian correlator to a two-layer pattern-matched heteroassociative circuit. Therefore, this class of networks possesses good application prospects in the areas of pattern recognition, signal and image processing, etc. In Kosko's investigation of the global stability of BAM models, the severe constraint of having a symmetric connection weight matrix had to be satisfied. However, from the viewpoint of biological neural networks and their implementations using very large scale integrated circuit or optical technology, it is impossible to maintain an absolutely symmetric connection weight matrix. Thus, investigation of the stability of BAM networks with asymmetric connections has been a focus of this field. Recently, a two-layer heteroassociative network, referred to as a BAM model with axonal signal transmission delays, was proposed by Gopalsamy and He [2] and by Liao *et al.* [3,4], which led to important advances in many fields such as pattern recognition, automatic control, and so on. The stability of this type of network has been extensively studied. Interested readers may refer to [2–8,17].

However, all the existing investigations of BAM models with delays were restricted to pure-delay models [2–8,17]. A hybrid model in which both instantaneous and delayed signaling occur [9–16] has not been studied for BAM neural networks. We also note that some papers are concerned only with the stability properties of pure-delay BAM models [2–7,17], without providing any information about the transient responses and decay rates (i.e., exponential convergence rates). To the best of our knowledge, a general delay-independent result concerning the properties of transient responses and convergence rates has not been applied to the hybrid BAM models with delays [2–8,17].

In this paper, the convergence dynamics of hybrid bidirectional associative memory neural networks with constant transmission delays is studied in detail. In the model, the synaptic connection weights are assumed asymmetric and the nonlinear activation functions are not necessarily differentiable, monotonic, or nondecreasing. By using Halanay-type inequalities, we obtain a set of easily verifiable delay-independent sufficient conditions for the hybrid BAM with delays to converge exponentially to the equilibria associated with temporally uniform external inputs introduced from outside the network. Our results are less conservative and restrictive than the previously known results, as illustrated by an example.

II. GLOBAL EXPONENTIAL STABILITY

We investigate the existence and exponential stability of the unique equilibrium of hybrid bidirectional associative memory neural networks described by

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{p} w_{ji} s_j(y_j(t)) + \sum_{j=1}^{p} w_{ji}^{\tau} s_j(y_j(t-\tau_{ji})) + I_i,\\ \frac{dy_j(t)}{dt} &= -b_j y_j(t) + \sum_{i=1}^{n} v_{ij} s_i(x_i(t)) + \sum_{i=1}^{n} v_{ij}^{\tau} s_i(x_i(t-\sigma_{ij})) + J_j\end{aligned} \tag{1}$$

for $i \in \{1,2,...,n\}$, $j \in \{1,2,...,p\}$, $t > 0$. In Eq. (1), $x_i(t)$ and $y_j(t)$ denote the membrane potentials of the ith neurons from the neuronal fields F_X and F_Y, respectively; $w_{ji}, w_{ji}^{\tau}, v_{ij}, v_{ij}^{\tau}$ denote the synaptic connection weights; I_i and J_j denote the external inputs to the neurons introduced from outside the network; $\sigma_{ij} > 0$ and $\tau_{ji} > 0$ denote the time delays required for neural processing and axonal transmission of signals; $a_i > 0$ and $b_j > 0$ denote the rates with which the ith neurons from the neuronal fields F_X and F_Y, respectively, reset their potentials to their resting states when isolated from other neurons and inputs; $s_i(\ )$ and $s_j(\ )$ denote the nonlinear activation functions. Some authors have studied a pure-delay model (with $w_{ji} = 0, v_{ij} = 0$). However, a hybrid model in

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which both instantaneous signaling and delayed signaling occur (with $w_{ji}\neq 0,\nu_{ij}\neq 0,w_{ji}^{\tau}\neq 0,v_{ij}^{\tau}\neq 0$) has not yet been investigated.

The nonlinear activation functions $s_i(\,)$ and $s_j(\,)$ are assumed to satisfy the following requirements:

$$s_i\,,s_j:R\to R,$$

$$|s_i(u)-s_i(v)|\leqslant L_i|u-v|,\quad |s_j(u)-s_j(v)|\leqslant M_j|u-v|, \tag{2}$$

$$|s_i(u)|\leqslant A_i<+\infty,\quad |s_j(u)|\leqslant B_j<+\infty$$

for $i\in\{1,2,...,n\},j\in\{1,2,...,p\},u,v\in R$, where $L_i>0$ and $M_j>0$ denote the Lipschitz constants and $A_i>0$, $B_j>0$ are constants. The functions $s_i(\,)$ and $s_j(\,)$ are defined on R with values in R.

The initial functions associated with the system (1) are given by

$$x_i(s)=\phi_i(s),\quad s\in[-\tau,0],\quad \tau=\max_{1\leqslant i\leqslant n}\{\tau_{ji}\},$$
$$y_j(s)=\varphi_j(s),\quad s\in[-\sigma,0],\quad \sigma=\max_{1\leqslant i\leqslant n}\{\sigma_{ij}\}, \tag{3}$$

where $\phi_i(\,)$ and $\varphi_j(\,)$ denote real-valued continuous functions defined on $[-\tau,\,0]$ and $[-\sigma,\,0]$, respectively.

An equilibrium of Eq. (1) is denoted by $\binom{x*}{y*}$, where $x*=[x_1^*,x_2^*,\ldots,x_n^*]^T$, $y*=[y_1^*,y_2^*,\ldots,y_p^*]^T$, and

$$x_i^*=\frac{1}{a_i}\left(\sum_{j=1}^{p}(w_{ji}+w_{ji}^{\tau})s_j(y_j^*)+I_i\right),\quad i\in\{1,2,...,n\},$$
$$y_j^*=\frac{1}{b_j}\left(\sum_{i=1}^{n}(v_{ij}+v_{ij}^{\tau})s_i(x_i^*)+J_j\right),\quad j\in\{1,2,...,p\}. \tag{4}$$

Under certain circumstances, one may want to estimate the rate of convergence of each or some of the neurons in the network. In this section, we employ Halanay-type inequalities (see [18]) to obtain another set of easily verifiable delay-independent sufficient conditions that ensure the global exponential stability of the equilibrium point. For the convenience of readers, here we provide a scalar version of the Halanay inequality discussed in [18].

Lemma: Halanay inequality. Let $v(t)>0$ for $t\in R$, $\tau\in[0,\infty]$, and $t_0\in R$. Suppose that

$$v'(t)\leqslant -av(t)+b\left(\sup_{t-\tau\leqslant s\leqslant t}v(s)\right)\quad \text{for } t>t_0. \tag{5}$$

If $a>b>0$, there exist constants $\gamma>0$ and $k>0$ such that $v(t)\leqslant ke^{-\gamma(t-t_0)}$ for $t>t_0$.

We first let

$$K_1=\max_{1\leqslant i\leqslant n}\left\{\sup_{s\in[-\tau,0]}|x_i(s)-x_i^*|\right\},$$
$$K_2=\max_{1\leqslant j\leqslant p}\left\{\sup_{s\in[-\sigma,0]}|y_j(s)-y_j^*|\right\}, \tag{6}$$

where either K_1 or K_2 is positive.

Theorem. If condition (3) is satisfied, and letting

$$a_i>\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|)M_j,\quad i\in\{1,2,...,n\},$$
$$b_j>\sum_{i=1}^{n}(|v_{ij}|+|v_{ij}^{\tau}|)L_i,\quad j\in\{1,2,...,p\}, \tag{7}$$

then the equilibrium $\binom{x*}{y*}$ of Eq. (1) is unique and, moreover, there exists a constant $\eta>0$ such that

$$|x_i(t)-x_i^*|\leqslant Ke^{-\eta t},\quad |y_j(t)-y_j^*|\leqslant Ke^{-\eta t} \tag{8}$$

for $i\in\{1,2,...,n\}$, $j\in\{1,2,...,p\}$, $t>0$, where $K=\max\{K_1,K_2\}>0$ and the constants K_1 and K_2 are defined above.

Proof. The proof of the existence of equilibria for Eq. (1) is similar to that presented in [4,17]. To establish the uniqueness of $\binom{x*}{y*}$, we proceed by supposing the existence of another equilibrium $\binom{u*}{v*}$ of Eq. (1). We claim that $x_i^*=u_i^*$ and $y_j^*=v_j^*$ componentwise for all $i\in\{1,2,...,n\}$ and $j\in\{1,2,...,p\}$. Suppose the claim is invalid in the sense that there exist some components of $x*$ and $u*$ (say x_k^* and u_k^*, respectively) such that $x_k^*\neq u_k^*$. By using Eq. (3), we obtain from Eq. (4) that

$$a_i|x_i^*-u_i^*|\leqslant\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|)M_j|y_j^*-v_j^*|,$$
$$i\in\{1,2,...,n\},$$

$$b_j|y_j^*-v_j^*|\leqslant\sum_{i=1}^{n}(|v_{ij}|+|v_{ij}^{\tau}|)L_i|x_i^*-u_i^*|,\quad j\in\{1,2,...,p\},$$

which then lead to

$$a_k|x_k^*-u_k^*|\leqslant 0,\quad 0\leqslant(|v_{kj}|+|v_{kj}^{\tau}|)L_k|x_k^*-u_k^*|,$$
$$j\in\{1,2,..,p\}.$$

This in turn contradicts the assumption that $x_k^*\neq u_k^*$. Hence, the claim holds and the uniqueness of the equilibrium is proved.

We can now prove the global exponential stability of $\binom{x*}{y*}$ of Eq. (1) under condition (7). We note that $\binom{x(t)}{y(t)}$ denotes an arbitrary solution of Eq. (1) for $t>0$. We consider the functions $P_i(\,)$ and $Q_i(\,)$ defined by

$$P_i(\mu_i)=a_i-\mu_i-\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|e^{\mu_i\tau_{ji}})M_j, \quad \mu_i\in[0,\infty)$$
$$Q_j(\nu_j)=b_j-\nu_j-\sum_{i=1}^{n}(|\nu_{ij}|+|\nu_{ij}^{\tau}|e^{\nu_j\sigma_{ij}})L_i, \quad \nu_j\in[0,\infty) \tag{9}$$

for $i\in\{1,2,...,n\}$, $j\in\{1,2,...,p\}$. We notice from condition (7) that

$$a_i-\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|)M_j\geqslant\xi \quad \text{for all} \quad i\in\{1,2,...,n\},$$
$$b_j-\sum_{i=1}^{n}(|v_{ij}|+|v_{ij}^{\tau}|)L_i\geqslant\xi \quad \text{for all} \quad j\in\{1,2,...,p\}, \tag{10}$$

where $\xi=\min\{\xi_1,\xi_2\}$ and

$$\xi_1=\min_{1\leqslant i\leqslant n}\left\{a_i-\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|)M_j\right\}>0,$$

$$\xi_2=\min_{1\leqslant j\leqslant p}\left\{b_j-\sum_{i=1}^{n}(|v_{ij}|+|v_{ij}^{\tau}|)L_i\right\}>0.$$

It follows from Eqs. (9) and (10) that $P_i(0)\geqslant\xi$ and $Q_j(0)\geqslant\xi$ for all $i\in\{1,2,...,n\}$, $j\in\{1,2,...,p\}$. We observe that $P_i(\mu_i)$ and $Q_j(v_j)$ are continuous for $\mu_i,\nu_j\in[0,\infty]$ and, moreover, $P_i(\mu_i),Q_j(\nu_i)\rightarrow-\infty$ as $\mu_i,\nu_j\rightarrow\infty$. Thus there exist constants $\tilde{\mu}_i,\tilde{\nu}_j\in(0,\infty)$ such that

$$P_i(\tilde{\mu}_i)=a_i-\tilde{\mu}_i-L_i\sum_{j=1}^{p}(|v_{ij}|+|v_{ij}^{\tau}|e^{\tilde{\mu}_i\sigma_{ij}})=0,$$
$$Q_j(\tilde{\nu}_j)=b_j-\tilde{\nu}_j-M_j\sum_{i=1}^{n}(|w_{ji}|+|w_{ji}^{\tau}|e^{\tilde{\nu}_j\tau_{ji}})=0 \tag{11}$$

for $i\in\{1,2,...,n\}$, $j\in\{1,2,...,p\}$. By choosing $\eta=\min\{\tilde{\mu}_1,\tilde{\mu}_2,...,\tilde{\mu}_n,\tilde{\nu}_1,\tilde{\nu}_2,...,\tilde{\nu}_p\}$, where obviously $\eta>0$, we obtain from Eq. (11) that

$$P_i(\eta)=a_i-\eta-\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|)M_je^{\eta\tau_{ji}}\geqslant 0,$$
$$Q_j(\eta)=b_j-\eta-\sum_{i=1}^{n}(|v_{ij}|+|v_{ij}^{\tau}|)L_ie^{\eta\sigma_{ij}}\geqslant 0 \tag{12}$$

for $i\in\{1,2,...,n\}$, $j\in\{1,2,...,p\}$. We define functions $X_i(\,)$ and $Y_j(\,)$ as follows:

$$X_i(t)=e^{\eta t}|x_i(t)-x_i^*|, \quad Y_j(t)=e^{\eta t}|y_j(t)-y_j^*|. \tag{13}$$

We then use Eq. (13) to derive a system of Halanay-type inequalities given by

$$\frac{d^+X_i(t)}{dt}\leqslant-(a_i-\eta)X_i(t)+\sum_{j=1}^{p}(|w_{ji}|+|w_{ji}^{\tau}|)M_je^{\eta\tau_{ji}}\left(\sup_{s\in[t-\tau,\tau]}Y_j(s)\right),$$
$$\frac{d^+Y_j(t)}{dt}\leqslant-(b_j-\eta)Y_j(t)+\sum_{i=1}^{n}(|v_{ij}|+|v_{ij}^{\tau}|)L_ie^{\eta\sigma_{ij}}\left(\sup_{s\in[t-\sigma,t]}X_i(s)\right). \tag{14}$$

We notice from Eq. (13) that $X_i(t)\leqslant K$ for all $i\in\{1,2,...,n\}$, $t\in[-\tau,0]$ and $Y_j(t)\leqslant K$ for all $j\in\{1,2,...,p\}$, $t\in[-\sigma,0]$, where $K>0$ is given by Eq. (8). We claim that

$$X_i(t)\leqslant K, \quad Y_j(t)\leqslant K \quad \text{for all} \quad i\in[1,2,...,n],$$
$$j\in[1,2,...,p], \quad t>0. \tag{15}$$

Suppose that this claim does not hold in the sense that there is one component among $X_i(\,)$ [say, $X_k(\,)$] and a first time $t_1>0$ such that

$$X_k(t)\leqslant K, \quad t\in[-\tau,t_1], \quad X_k(t_1)=K, \quad \frac{d^+X_k(t_1)}{dt}>0, \tag{16}$$

while

$$X_i(t)\leqslant K, \quad i\neq k, \quad t\in[-\tau,t_1];$$
$$Y_j(t)\leqslant K, \quad t\in[-\sigma,t_1]. \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (14), we obtain

$$0<\frac{d^+X_k(t_1)}{dt}\leqslant-\left(a_k-\eta+\sum_{j=1}^{p}(|w_{jk}|+|w_{ji}^{\tau}|e^{\eta\tau_{ji}})M_j\right)K,$$

and by applying Eq. (12) to the above inequality we obtain $0<d^+X_k(t_1)/dt\leqslant 0$, which means a contradiction. Consequently, the claim (15) must hold. We put Eq. (13) into Eq. (15) to obtain Eq. (8), which asserts the global exponential stability of $\binom{x^*}{y^*}$ of Eq. (1). The proof is completed. ■

Example. Consider the following model:

$$\dot{x}_1(t)=-2x_1(t)+0.5\tanh\left(\frac{2}{\sqrt{3}}y_1(t-\tau)\right)+0.5\tanh\left(\frac{2}{\sqrt{3}}y_2(t-\tau)\right),$$

$$\dot{x}_2(t)=-2x_2(t)-0.5\tanh\left(\frac{2}{\sqrt{3}}y_1(t-\tau)\right)+0.5\tanh\left(\frac{2}{\sqrt{3}}y_2(t-\tau)\right), \tag{18}$$

$$\dot{y}_1(t) = -2y_1(t) + 0.25\tanh\left(\frac{2}{\sqrt{3}}x_1(t-\sigma)\right) - 0.25\tanh\left(\frac{2}{\sqrt{3}}x_2(t-\sigma)\right),$$

$$\dot{y}_2(t) = -2y_2(t) + 0.25\tanh\left(\frac{2}{\sqrt{3}}x_1(t-\sigma)\right) + 0.25\tanh\left(\frac{2}{\sqrt{3}}x_2(t-\sigma)\right).$$

We can easily obtain $M_i\Sigma_{j=1}^{n}|w_{ji}^{\tau}| = 2/\sqrt{3} > 1$. Hence, the condition in [2] is not satisfied. Let $\lambda_i = \lambda_{n+j} = 1$, then the condition in [4] is also unsatisfied. It is easy to obtain

$$\Omega = \begin{pmatrix} \frac{4}{\sqrt{3}} & 0 & -0.5 & 0.5 \\ 0 & \frac{4}{\sqrt{3}} & -0.5 & -0.5 \\ -0.25 & 0.25 & \frac{2}{\sqrt{3}} & 0 \\ -0.25 & -0.25 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix},$$

which is not an M matrix. Hence, the conditions of [7] are not satisfied. We can compute $\min\{b_j - M_j\Sigma_{j=1}^{2}|w_{ji}^{\tau}|\} = 1 - 2/\sqrt{3} < 0$, then the condition in [6] does not hold. If we let $\lambda_i = 2/\sqrt{3}$, $\lambda_{n+j} = 20/\sqrt{3}$, $r_1 = r_2 = 0.5$ in [17], then we can compute $\delta_1 = 3 > 2$, $\varepsilon_1 = \eta_1 = 2\sqrt{3} > 2$. Hence, the conditions in [17] are not satisfied. However, we have

$$2 > \frac{2}{\sqrt{3}}(|w_{11}^{\tau}| + |w_{21}^{\tau}|) = \frac{2}{\sqrt{3}}, \quad 2 > \frac{2}{\sqrt{3}}(|w_{12}^{\tau}| + |w_{22}^{\tau}|) = \frac{2}{\sqrt{3}},$$

$$1 > \frac{2}{\sqrt{3}}(|v_{11}^{\tau}| + |v_{21}^{\tau}|) = \frac{1}{\sqrt{3}}, \quad 1 > \frac{2}{\sqrt{3}}(|v_{12}^{\tau}| + |w_{22}^{\tau}|) = \frac{1}{\sqrt{3}}.$$

Therefore, by the theorem, the system (18) has a globally exponentially stable equilibrium.

III. CONCLUSIONS

In this paper, we considered the hybrid BAM with time delays and obtained a set of easily verifiable delay-independent sufficient conditions for global exponential stability. The results obtained generalize and improve those of [2–8,17]. Without the constraints of symmetry of the connection weights and differentiability and monotonicity of the activation functions, we are provided with more freedom in designing the circuitry of hybrid bidirectional BAM neural networks for certain computational tasks. The results for global exponential stability [see the inequality (8)] can provide us with relevant estimates on how fast such networks can perform during real-time computations. We remark that, in the circuits of hybrid bidirectional BAM neural networks, the parameters a_i, b_j are controlled reciprocally by resistors. Hence, according to the condition (7), reducing the amount of resistance will increase the convergence rate (i.e., the performance) of the networks.

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